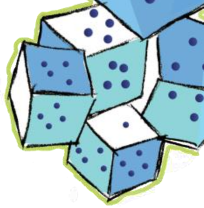
Illustration by Chris Brigman



Sandia National Laboratories

Practical Leverage-Based Sampling for Low-Rank Tensor Decomposition

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Joint work with
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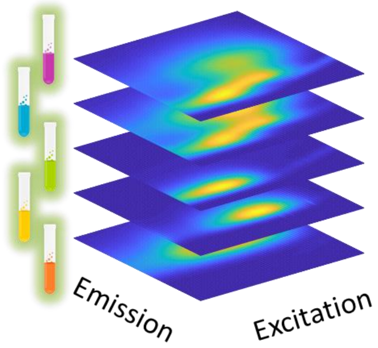
Supported by the DOE Office of Science Advanced Scientific Computing Research (ASCR) Applied Mathematics. Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

U.S. DEPARTMENT OF ENERGY

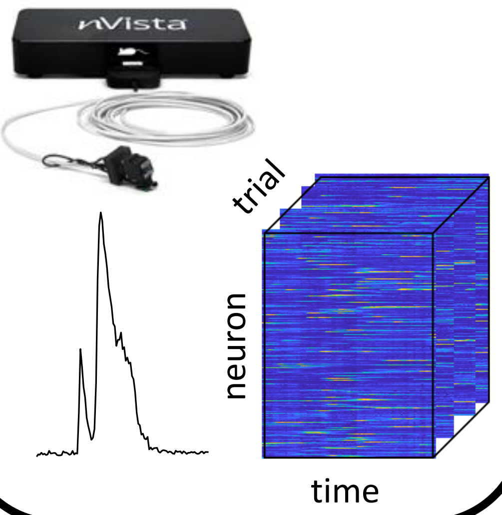
Tensors Come From Many Applications

- **Chemometrics:** Emission x Excitation x Samples (Fluorescence Spectroscopy)
- **Neuroscience:** Neuron x Time x Trial
- **Criminology:** Day x Hour x Location x Crime (Chicago Crime Reports)
- **Machine Learning:** Multivariate Gaussian Mixture Models Higher-Order Moments
- **Transportation:** Pickup x Dropoff x Time (Taxis)
- **Sports:** Player x Statistic x Season (Basketball)
- **Cyber-Traffic:** IP x IP x Port x Time
- **Social Network:** Person x Person x Time x Interaction-Type
- **Signal Processing:** Sensor x Frequency x Time
- **Trending Co-occurrence:** Term A x Term B x Time

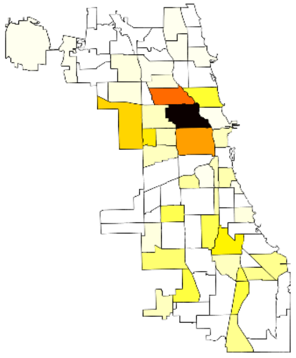
Chemometrics



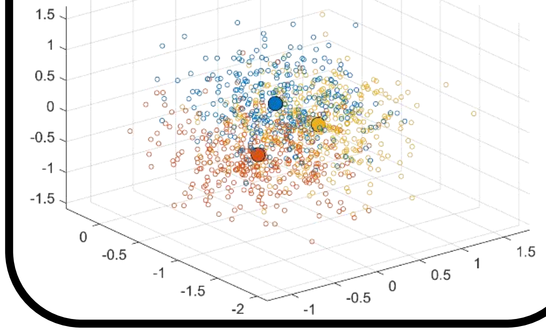
Neuroscience



Criminology



Machine Learning

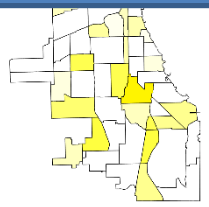


Tensors Come From Many Applications

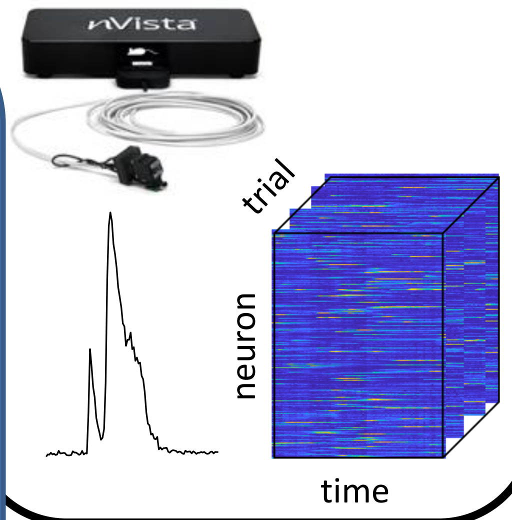
- **Chemometrics:** Emission x Excitation x Samples (Fluorescence Spectroscopy)
- **Neuroscience:** N
- **Criminology:** D
  (Chicago Crime
- **Machine Learn**
  Mixture Model
- **Transportation**
- **Sports:** Player
- **Cyber-Traffic:** I
- **Social Network**
  Interaction-Type
- **Signal Processing:** Sensor x Frequency x Time
- **Trending Co-occurrence:** Term A x Term B x Time

Tensor Decomposition Finds Patterns in Massive Data (Unsupervised Learning)

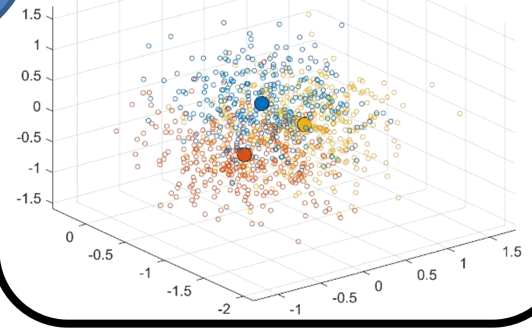
Chemometrics



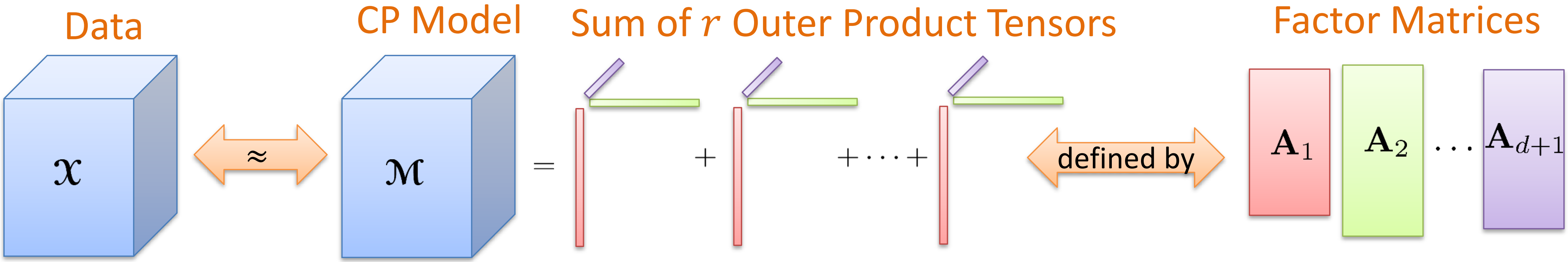
Neuroscience



Machine Learning



Tensor Decomposition Identifies Factors



$\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d+1}}$

$\mathcal{M} = [\![\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{d+1}]\!] \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d+1}}$

$\mathbf{A}_k \in \mathbb{R}^{n_k \times r}$

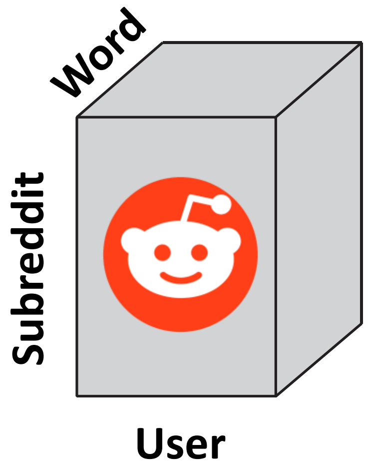
$x_i = x(i_1, i_2, \ldots, i_{d+1})$

$$m_i = m(i_1, i_2, \ldots, i_{d+1}) = \sum_{j=1}^{r} \prod_{k=1}^{d+1} a_k(i_k, j)$$

Model Rank

Example Sparse Multiway Data: Reddit

- Reddit is an American social news aggregator, web content rating, and discussion website
  - A “subreddit” is a discussion forum on a particular topic
- Tensor obtained from frost.io (http://frostt.io/tensors/reddit-2015/)
  - Built from reddit comments posted in the year 2015
  - Users and words with less than 5 entries have been removed



Reddit Tensor

8 million users
200 thousand subreddits
8 million words

4.7 billion non-zeros ($10^{-8}\%$)
106 gigabytes

$x(i,j,k) =$ log (1 + the number of times user i used word j in subreddit k)

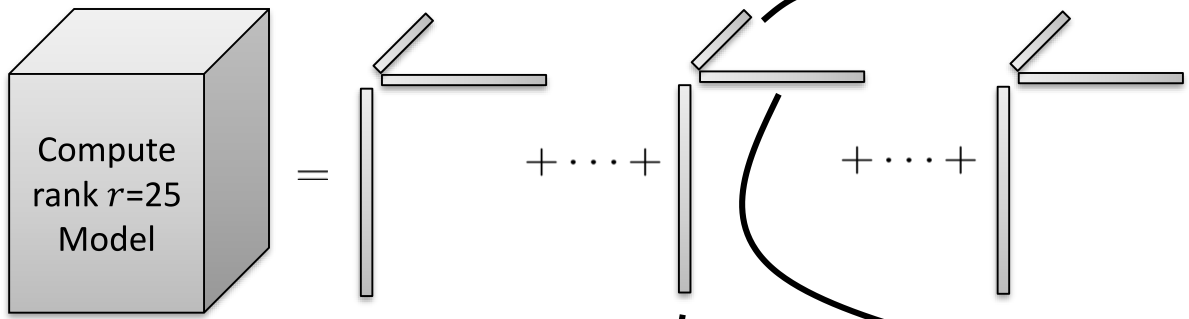
Used a rank $\boldsymbol{r = 25}$ decompsition

Smith et al (2017). “FROSTT: The Formidable Open Repository of Sparse Tensors and Tools”

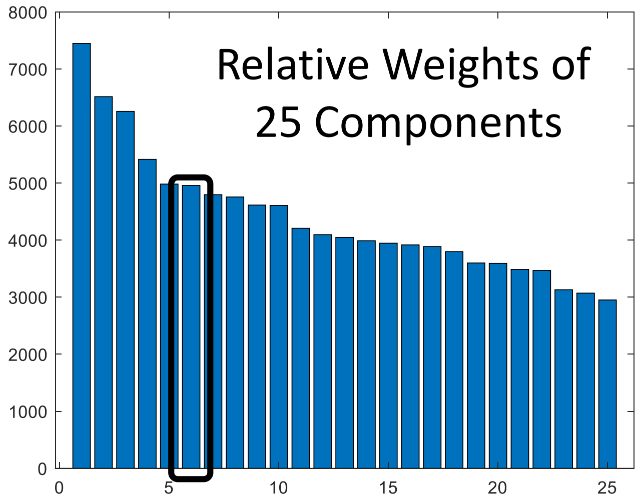
Interpreting Reddit Components

Reddit Tensor

Component 6

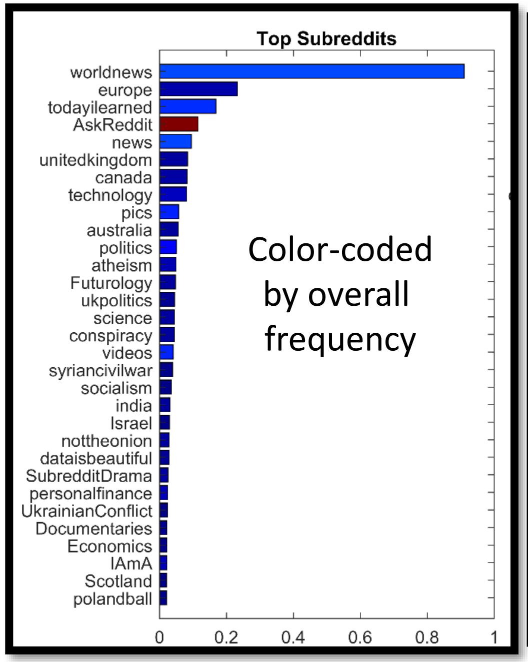


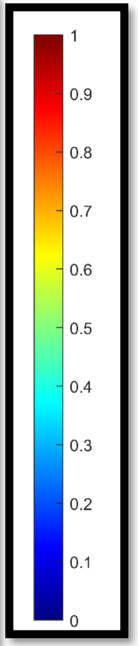
variance explained ≈ 6%

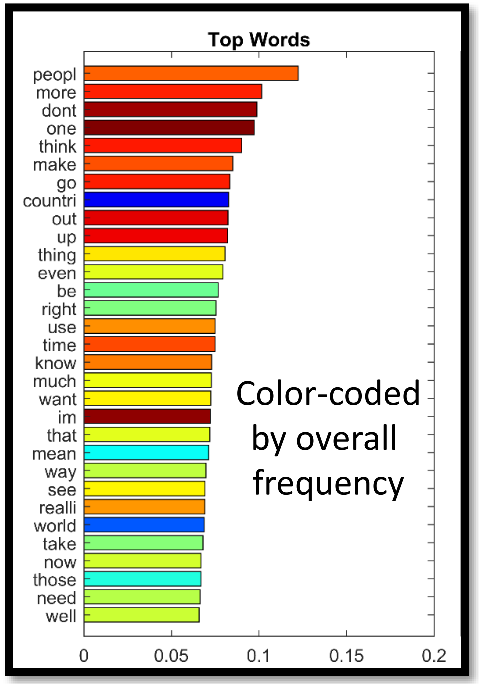


Largest entries,
in absolute value

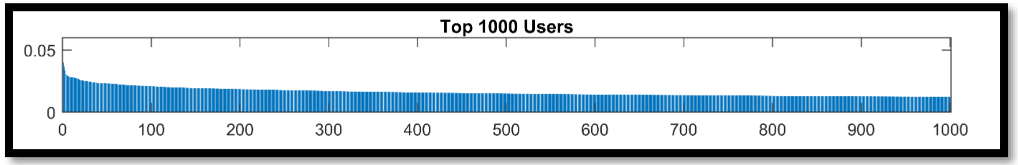
Largest of 200K
entries,
in absolute value





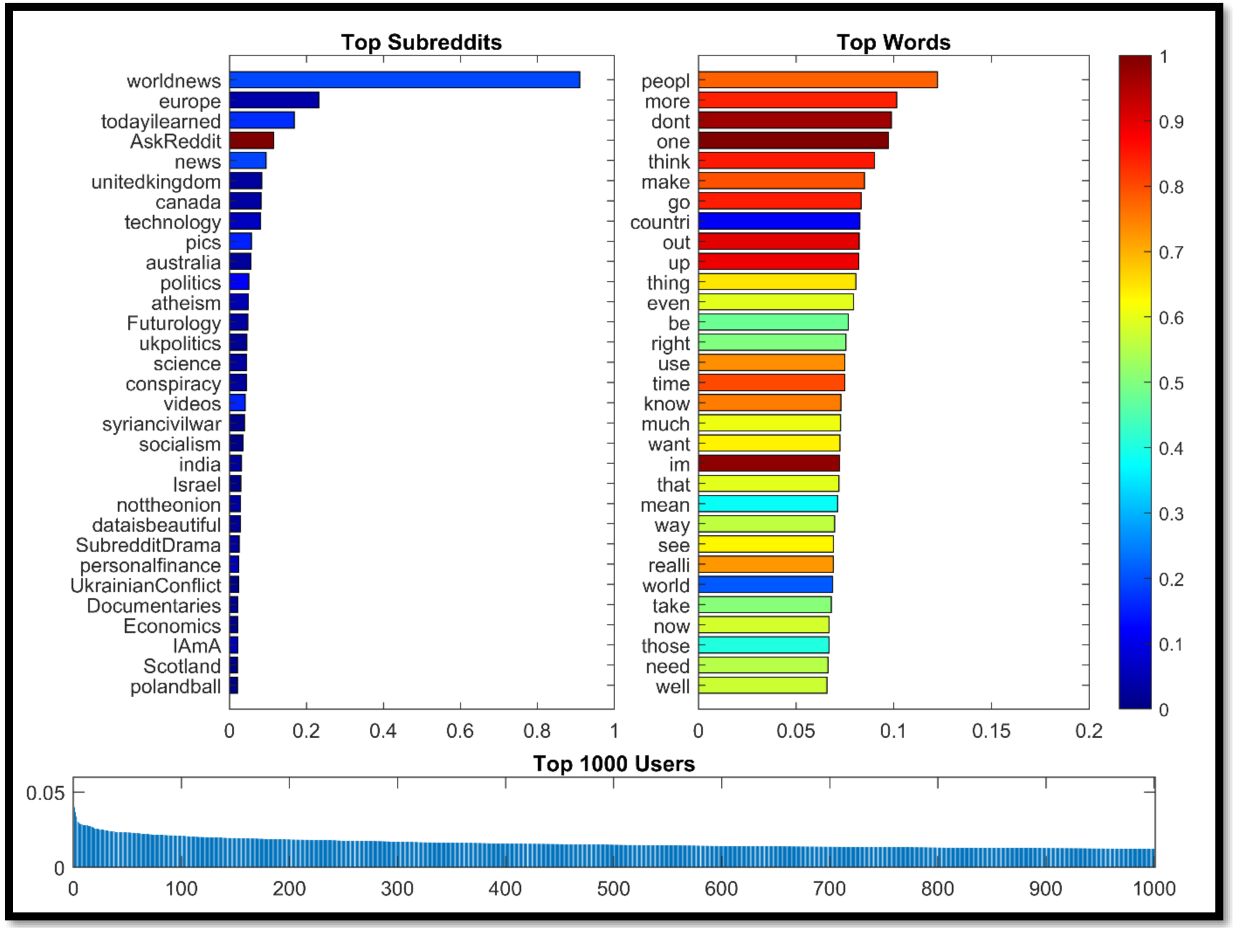


Largest of 8M entries,
in absolute value

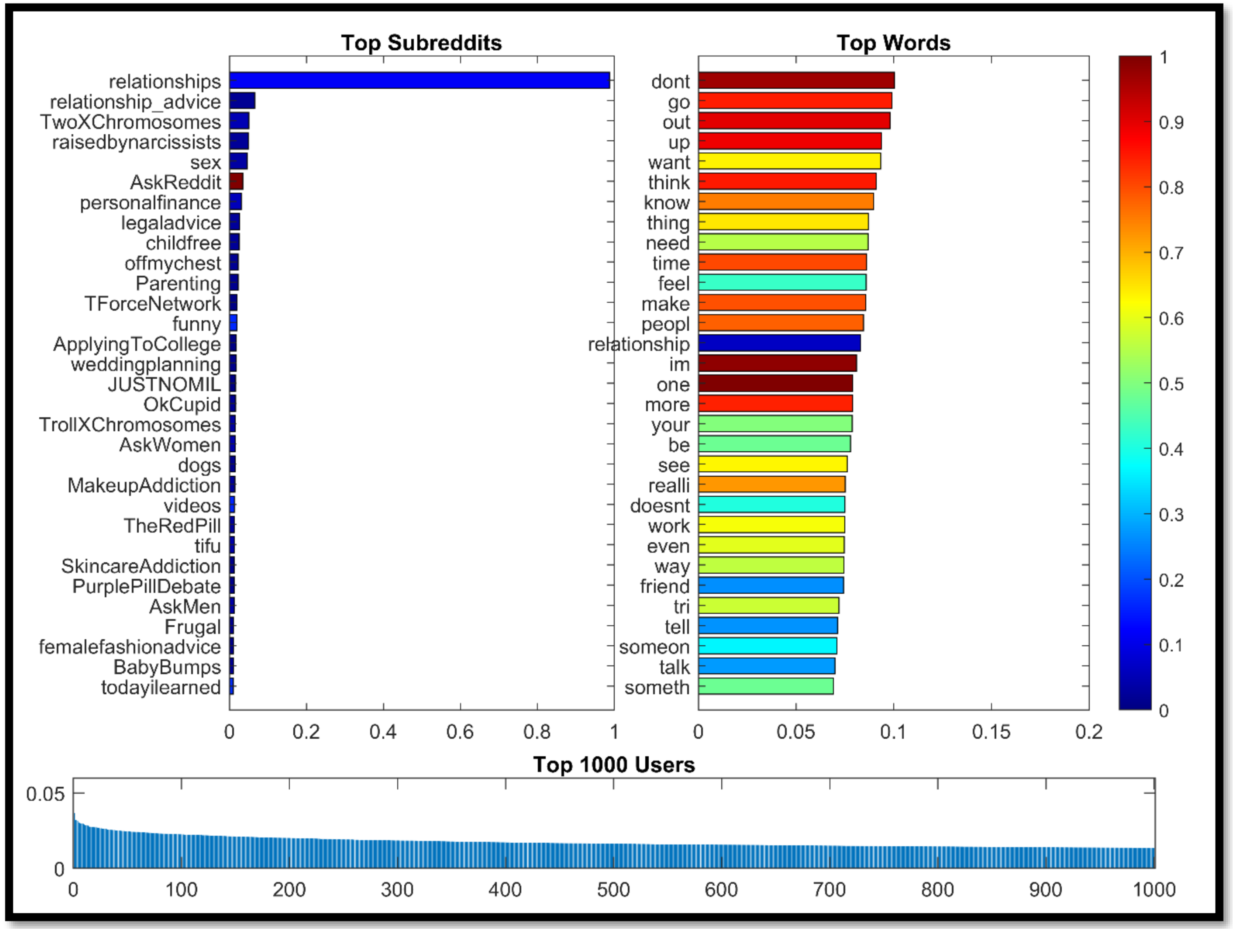


Example Reddit Components Include Rare Words Apropos to High-Scoring Reddits

Component #6: International News

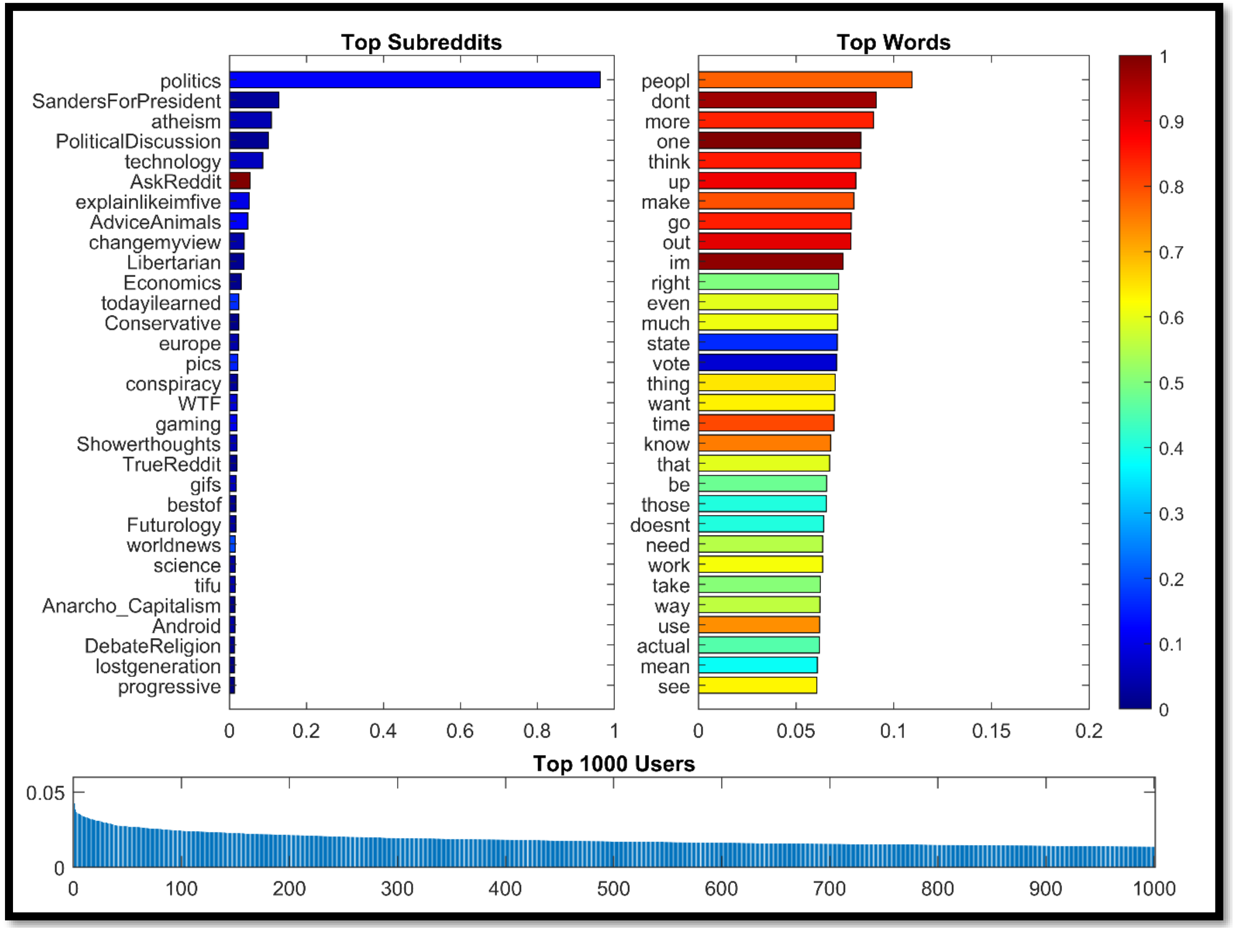


Component #8: Relationships

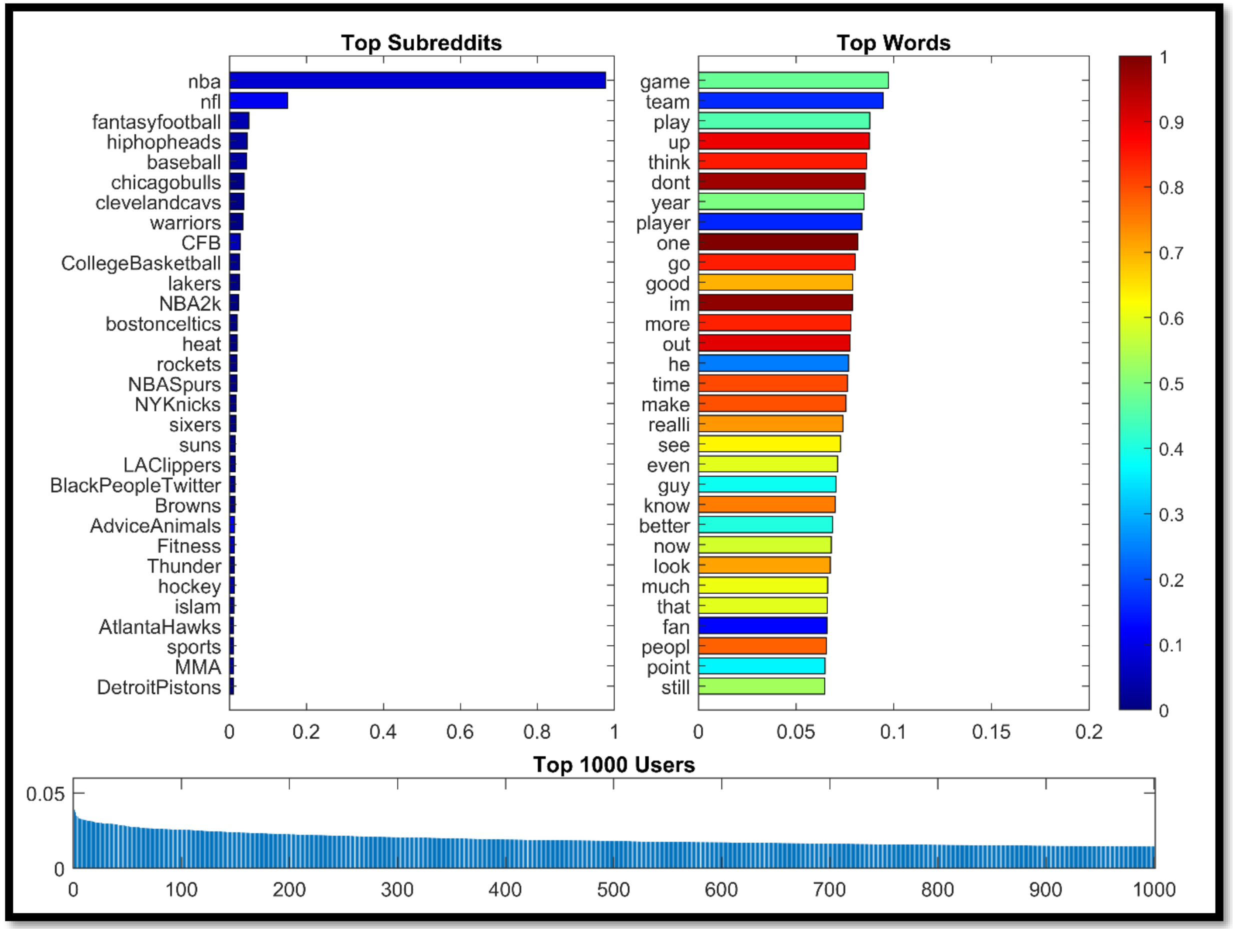


Example Reddit Components Include Rare Words Apropos to High-Scoring Reddits

Component #9: U.S. Politics (2015)

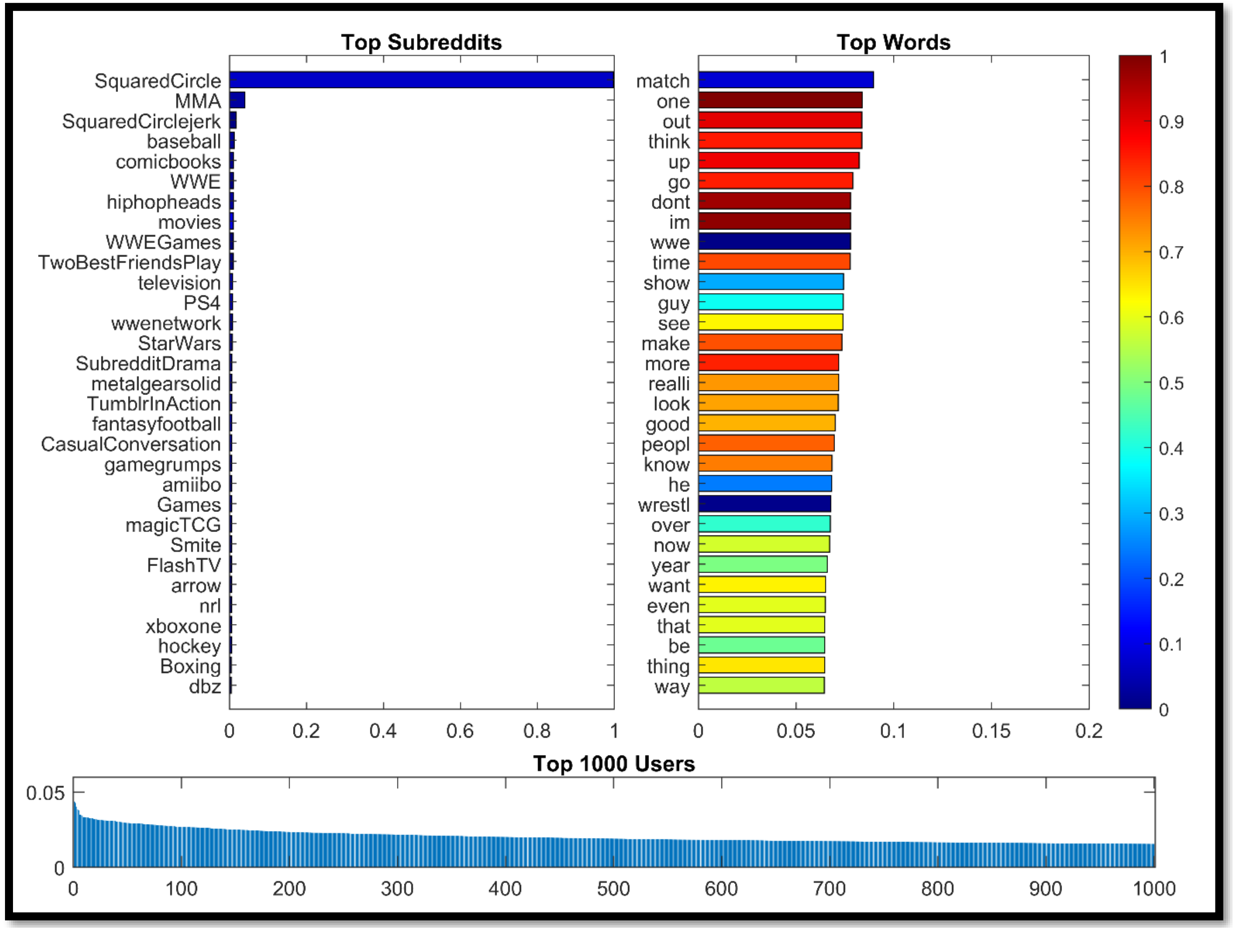


Component #11: Sports

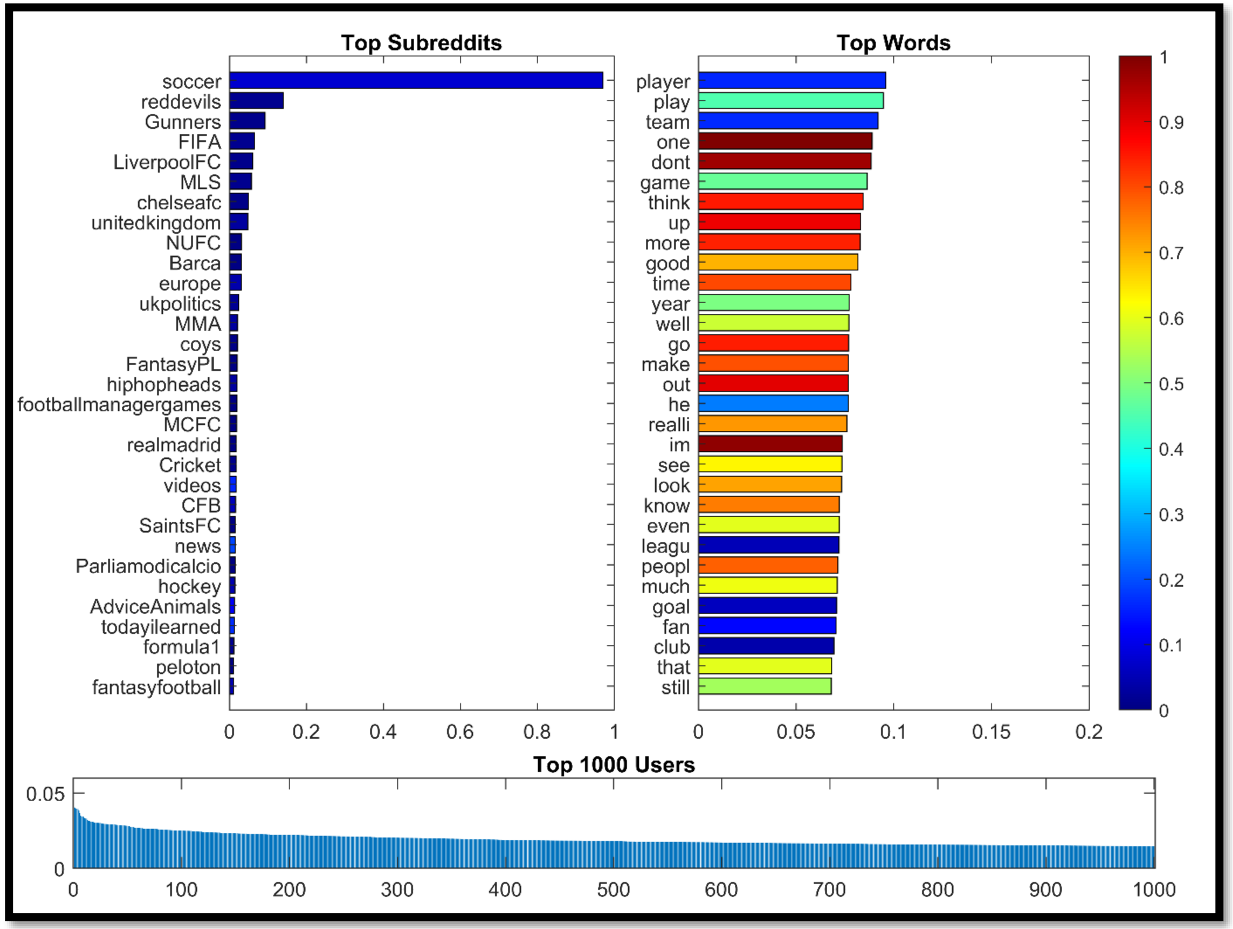


Example Reddit Components Include Rare Words Apropos to High-Scoring Reddits

Component #15: Wrestling

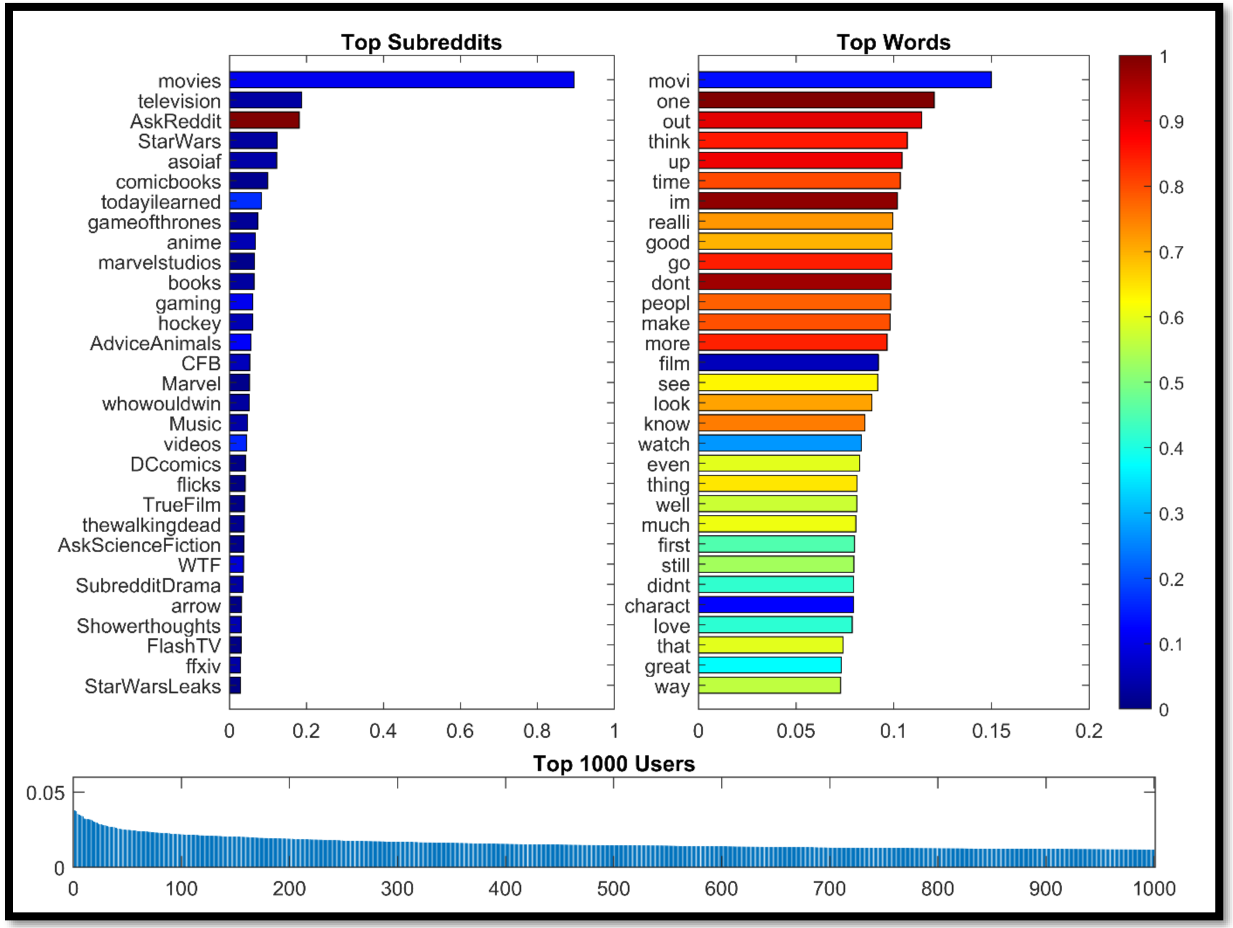


Component #18: Soccer

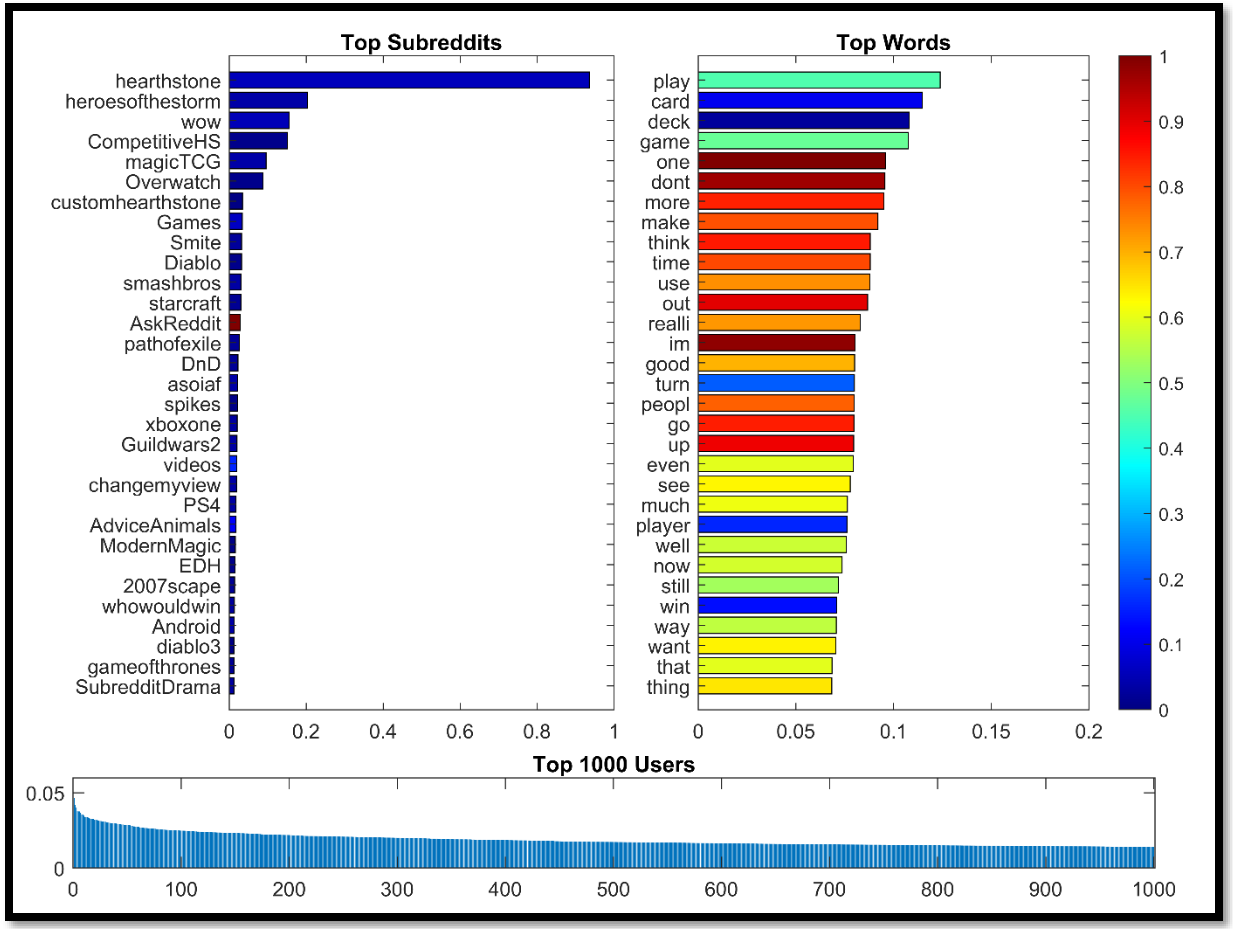


Example Reddit Components Include Rare Words Apropos to High-Scoring Reddits

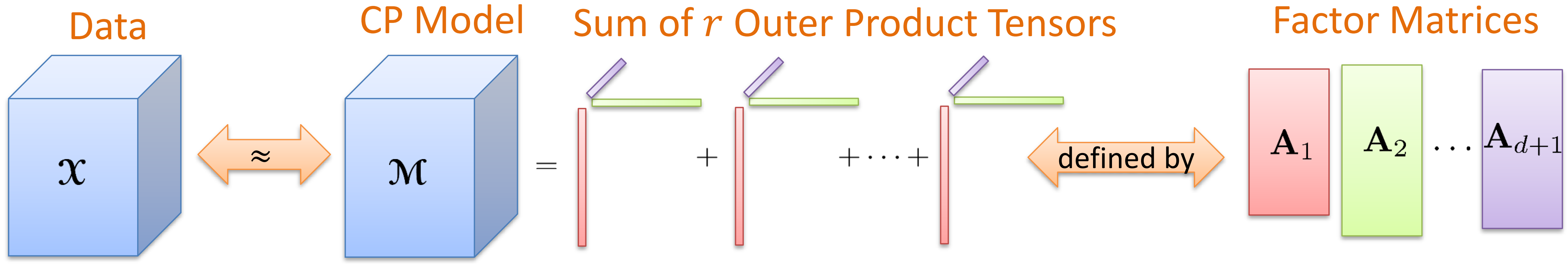
Component #19: Movies & TV



Component #18: Computer Card Game



Tensor Decomposition Identifies Factors



$$\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d+1}} \qquad \mathcal{M} = [\![\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{d+1}]\!] \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d+1}} \qquad \mathbf{A}_k \in \mathbb{R}^{n_k \times r}$$

$$x_i = x(i_1, i_2, \ldots, i_{d+1}) \qquad m_i = m(i_1, i_2, \ldots, i_{d+1}) = \sum_{j=1}^{r} \prod_{k=1}^{d+1} a_k(i_k, j)$$

Model Rank

<u>Key Idea</u>: Alternate among the d factor matrices, fixing all but that one and solving. Each subproblem is linear least squares.

Prototypical CP Least Squares Problem has Khatri-Rao Product (KRP) Structure

$$\min_{\mathbf{B}} \|\mathbf{Z}\mathbf{B}^{\intercal} - \mathbf{X}^{\intercal}\|^2$$

$N \gg r, n$

Linking back to mode-(d+1) least squares subproblem

$$N = \prod_{k=1}^{d} n_k$$

$$n = n_{d+1}$$

$\mathbf{Z} \in \mathbb{R}^{N \times r}$: Khatri-Rao Product (KRP) Structure

$\mathbf{B}^{\intercal} \in \mathbb{R}^{r \times n}$: Unknown

$-$

$\mathbf{X}^{\intercal} \in \mathbb{R}^{N \times n}$: May Be Very Sparse

$\mathbf{B} = \mathbf{A}_{d+1}$

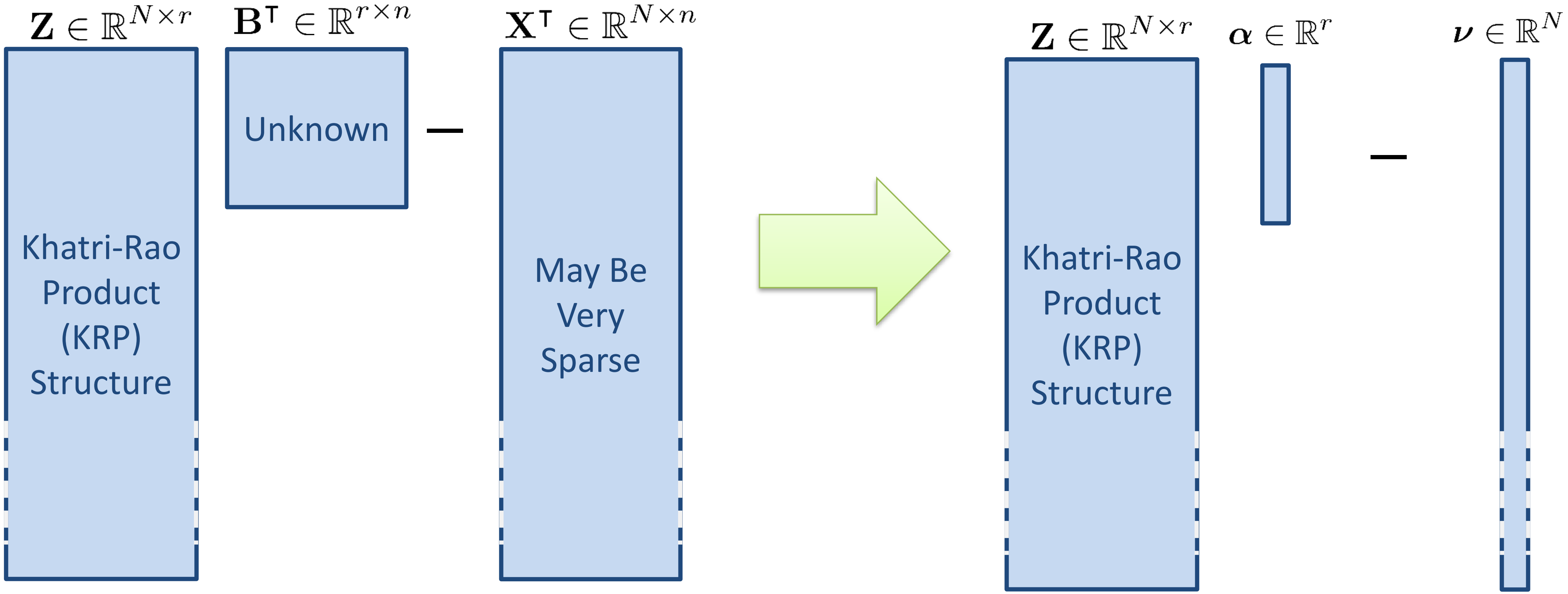
$\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_1$

$\mathbf{X} = \mathbf{X}_{(d+1)}$

- KRP costs $O(Nr)$ to form
- System costs $O(Nnr^2)$ to solve
- KRP structure
  - Cost reduced to $O(Nnr)$
- KRP structure + data sparse
  - Cost reduced to $O\big(r\,\mathrm{nnz}(\mathbf{X})\big)$

For Ease of Discussion: Simplify KRP Least Squares to Single Right-Hand Side

$$\min_{\mathbf{B}} \|\mathbf{Z}\mathbf{B}^\intercal - \mathbf{X}^\intercal\|^2 \qquad\qquad \min_{\boldsymbol{\alpha}\in\mathbb{R}^r} \|\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\nu}\|^2$$

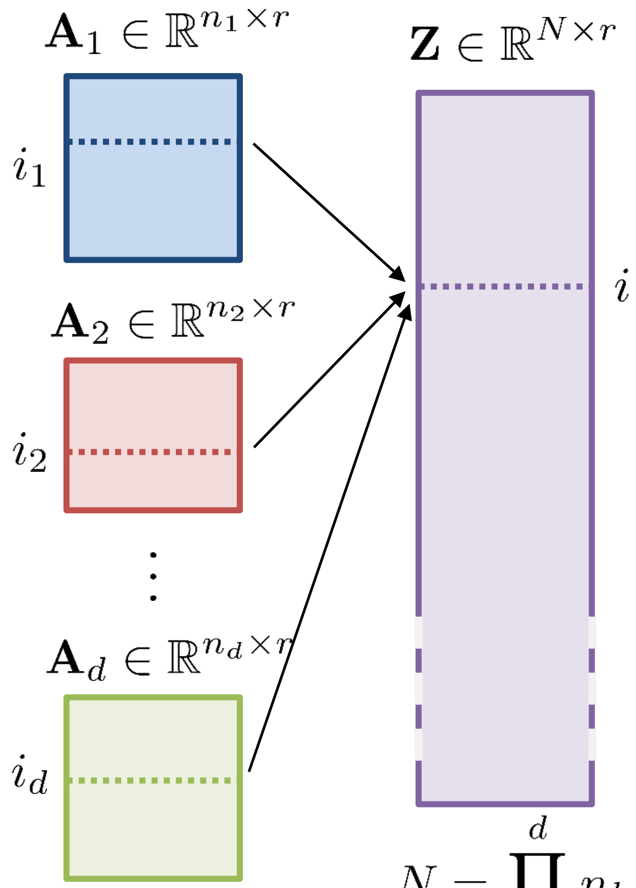


$N \gg r, n \qquad\qquad N \gg r$

Structure of Khatri-Rao Product (KRP): Hadamard Combinations of Rows of Inputs

KRP of d Matrices: $\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_1$

Number of columns is the same in all input matrices, but number of rows varies



$$N = \prod_{k=1}^{d} n_k$$

Each row of KRP is Hadamard product of specific rows in Factor Matrices:

$$\mathbf{Z}(i,:) = \mathbf{A}_1(i_1,:) * \cdots * \mathbf{A}_d(i_d,:)$$

where

$$i = (n_{d-1} \cdots n_1)(i_d - 1) + (n_{d-2} \cdots n_1)(i_{d-1} - 1) + \cdots + n_1(i_2 - 1) + i_1 \quad \in [N]$$

1-1 Correspondence between *linear index and multi index:*

$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

Ingredient #1: Sample Subset of Rows in Overdetermined Least Squares System

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^r} \|\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\nu}\|^2$$

$\mathbf{Z} \in \mathbb{R}^{N \times r}$ $\boldsymbol{\alpha} \in \mathbb{R}^r$ $-$ $\boldsymbol{\nu} \in \mathbb{R}^N$

Khatri-Rao Product (KRP) Structure

$N \gg r$

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^r} \|\boldsymbol{\Omega}\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\Omega}\boldsymbol{\nu}\|^2$$

$\boldsymbol{\Omega}\mathbf{Z} \in \mathbb{R}^{s \times r}$ $\boldsymbol{\alpha} \in \mathbb{R}^r$ $-$ $\boldsymbol{\Omega}\boldsymbol{\nu} \in \mathbb{R}^s$

Sampled KRP

Complexity reduced from $O(Nr)$ to $O(sr^2)$

Key surveys:
M. W. Mahoney, ***Randomized Algorithms for Matrices and Data***, 2011;
D. P. Woodruff, ***Sketching as a Tool for Numerical Linear Algebra***, 2014

How sample so that solution of sampled problem yields something close to the optimal residual of the original problem?

Ingredient #2: Weight Sampled Rows by Probability of Selection to Eliminate Bias

Probability distribution on rows of linear system

$$\sum_{i=1}^{N} p_i = 1$$

Not specifying yet how these probabilities are selected

Pick a single random index ξ with probability p_ξ

Choose

$$\mathbf{\Omega} = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{p_\xi}} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times 1}$$

ξth entry

Then (assuming all p_i positive) the sampled the sampled residual equals true residual in expectation:

$$\mathbb{E}\|\mathbf{\Omega Z}\boldsymbol{\alpha} - \mathbf{\Omega}\boldsymbol{\nu}\|^2 = \sum_{i=1}^{N} p_i \left(\left\| \frac{1}{\sqrt{p_i}} \mathbf{Z}(i,:)\boldsymbol{\alpha} - \frac{1}{\sqrt{p_i}} \nu_i \right\|^2 \right) = \|\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\nu}\|^2$$

Pick a s random indices ξ_j (with replacement) such that $P(\xi_j = i) = p_i$.

Choose $\mathbf{\Omega} \in \mathbb{R}^{s \times N}$ such that

Not specifying yet how s is determined

$$\omega(j,i) = \begin{cases} \frac{1}{\sqrt{s p_i}} & \text{if } \xi_j = i \\ 0 & \text{otherwise} \end{cases}$$

Each row has a single nonzero!

Then, as before, we have:

$$\mathbb{E}\|\mathbf{\Omega Z}\boldsymbol{\alpha} - \mathbf{\Omega}\boldsymbol{\nu}\|^2 = \|\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\nu}\|^2$$

Survey: D. P. Woodruff, ***Sketching as a Tool for Numerical Linear Algebra***, 2014

Theory Review: Connecting Probabilities, Leverage Scores, and Number of Samples

Given linear system: $\|\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\nu}\|^2 \text{ with } \mathbf{Z} \in \mathbb{R}^{N \times r}, \boldsymbol{\nu} \in \mathbb{R}^N$

And random sampling matrix: Pick a s random indices ξ_j such that $P(\xi_j = i) = p_i$ and define

$$\boldsymbol{\Omega} \in \mathbb{R}^{s \times N} \text{ with } \omega(j,i) = \begin{cases} \frac{1}{\sqrt{sp_i}} & \text{if } \xi_j = i \\ 0 & \text{otherwise} \end{cases}$$

Solve sampled problem: $\tilde{\boldsymbol{\alpha}}_* \equiv \arg\min_{\boldsymbol{\alpha} \in \mathbb{R}^r} \|\boldsymbol{\Omega}\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\Omega}\boldsymbol{\nu}\|_2^2$

Get probabilistic error bound: For error $\epsilon \in (0,1)$, confidence $1 - \delta \in (0,1)$, we have

$$P\left(\|\mathbf{Z}\tilde{\boldsymbol{\alpha}}_* - \boldsymbol{\nu}\|_2^2 \leq (1 + O(\epsilon))\|\mathbf{Z}\boldsymbol{\alpha}_* - \boldsymbol{\nu}\|_2^2\right) > 1 - \delta$$

when number of samples satisfies: $s = O(\epsilon^{-2} \ln\left(\frac{r}{\delta}\right) r\beta^{-1})$

where β-term: $\beta = \min_{i \in [N]} \frac{r\, p_i}{\ell_i(\mathbf{Z})} \in (0,1]$

We get to pick!

Leverage score

Want β as big as possible!

Ideally, $p_i = \ell_i(\mathbf{Z})/r$ for all $i \in [N]$, but …

A. Eshragh, et al., ***LSAR: Efficient Leverage Score Sampling Algorithm for the Analysis of Big Time Series Data***, arXiv:1911.12321, 2019;
D. P. Woodruff, ***Sketching as a Tool for Numerical Linear Algebra***, 2014

Ingredient #3: Leverage Scores Key to Limiting Samples (but too Expensive to Compute)

$$\begin{bmatrix} \boxed{z_{11} \quad z_{12} \quad \cdots \quad z_{1r}} \\ 0 \quad z_{22} \quad \cdots \quad z_{2r} \\ 0 \quad z_{32} \quad \cdots \quad z_{3r} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ 0 \quad z_{N2} \quad \cdots \quad z_{Nr} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_N \end{bmatrix}$$

$\ell_1(\mathbf{Z}) = 1$

$$\mathbf{Z} \in \mathbb{R}^{N \times r}$$

Leverage score:

Let $\mathbf{Q}$ be any orthonormal basis of the column space of $\mathbf{Z}$.

Leverage score of row i:

$$\ell_i(\mathbf{Z}) = \|\mathbf{Q}(i,:)\|_2^2 \in [0,1]$$

Coherence:

$$\mu(\mathbf{Z}) = \max_{i \in [N]} \ell_i(\mathbf{Z})$$

$$r/N \leq \mu(\mathbf{Z}) \leq 1$$

Rough Intuition:

Key rows have high leverage score

What if we do uniform sampling?

$p_i = \frac{1}{N}$ for all $i \in [N]$,

$$\beta = \min_{i \in [N]} \frac{r\, p_i}{\ell_i(\mathbf{Z})} = \min_{i \in [N]} \frac{r/N}{\ell_i(\mathbf{Z})}$$

$$s = O(\epsilon^{-2} \ln(r)\, r \beta^{-1})$$

Case 1: $\mu(\mathbf{Z}) = r/N$ (incoherent)

$$\Rightarrow \beta = 1 \Rightarrow s = O(\epsilon^{-2} \ln(r)\, r)$$

Case 2: $\mu(\mathbf{Z}) = 1$ (coherent)

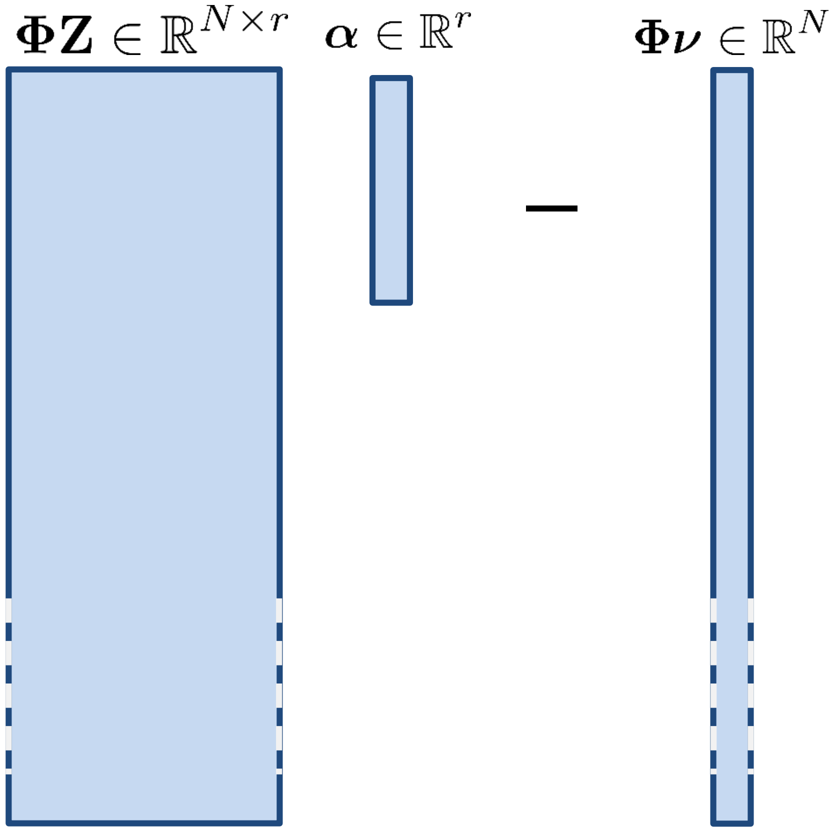
$$\Rightarrow \beta = r/N \Rightarrow s = O(\epsilon^{-2} \ln(r)\, N)$$

In Case 2, prefer $p_i = \ell_i(\mathbf{Z})/r$, <u>but</u> costs $O(Nr^2)$ to compute leverage scores!

Survey: D. P. Woodruff, ***Sketching as a Tool for Numerical Linear Algebra***, 2014

Aside: Uniform Sampling Okay for "Mixed" Dense Tensors (Inapplicable to Sparse)

Transform System: $\min_{\boldsymbol{\alpha} \in \mathbb{R}^r} \|\boldsymbol{\Phi}\mathbf{Z}\boldsymbol{\alpha} - \boldsymbol{\Phi}\boldsymbol{\nu}\|^2$



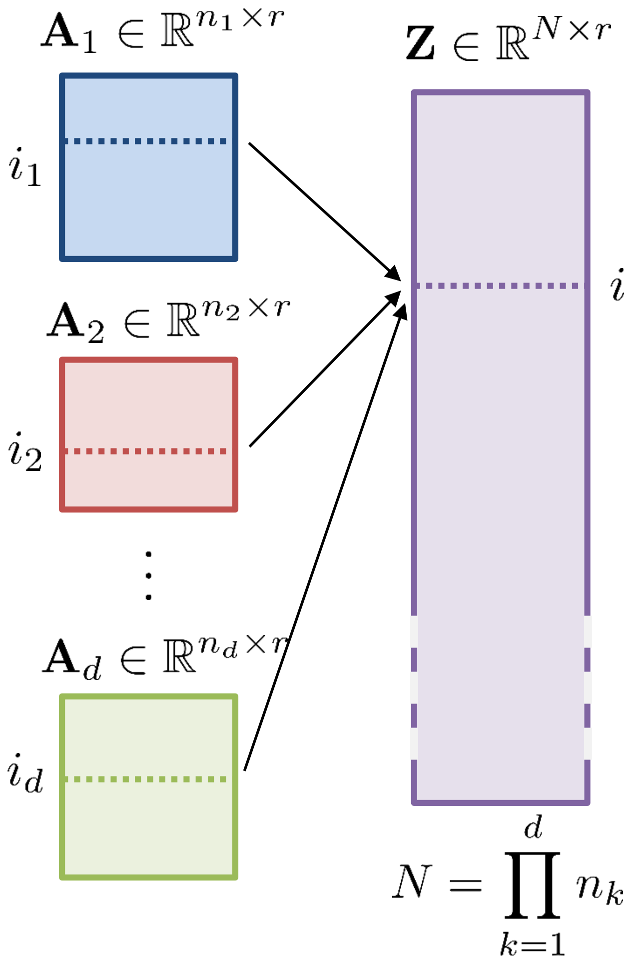
$N \gg r$

- Choose $\boldsymbol{\Phi}$ so that all leverage scores of $\boldsymbol{\Phi}\mathbf{Z}$ approximately equal, then uniform sampling yields $\beta \approx 1$
  - "Uniformize" the leverage scores per Mahoney
  - Fast Johnson-Lindenstrauss Transform (FJLT) uses random rows of matrix transformed by FFT and Rademacher diagonal
  - FJLT cost per iteration: $O(rN \log N)$
- Gaining Efficiency for KRP matrices
  - Transform individual factor matrices *before* forming $\mathbf{Z}$
  - Sample rows of $\mathbf{Z}$ implicitly
  - Kronecker Fast Johnson-Lindenstrauss Transform (KFJLT)
  - Special handling of right-hand side with preprocessing costs
  - KFJLT cost per iteration: $O(r \sum_k n_k \log n_k + sr^2)$
- References

  - C. Battaglino, G. Ballard, T. G. Kolda. **A Practical Randomized CP Tensor Decomposition**. *SIAM Journal on Matrix Analysis and Applications*, Vol. 39, No. 2, pp. 876-901, 26 pages, 2018. https://doi.org/10.1137/17M1112303
  - R. Jin, T. G. Kolda, R. Ward. **Faster Johnson-Lindenstrauss Transforms via Kronecker Products,** 2019. http://arxiv.org/abs/1909.04801

Ingredient #4: Exploit KRP Structure to Bound Leverage Scores

KRP: $\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_1$



Upper Bound on Leverage Score

Lemma (Cheng et al., NIPS 2016; Battaglino et al., SIMAX 2018):

$$\ell_i(\mathbf{Z}) \leq \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

Too expensive to calculate $O(Nr^2)$

Cheap to calculate individual leverage scores $O(r^2 \sum_k n_k)$

Set probability of sampling row i to:

$$p_i \equiv \frac{1}{r^d} \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

Tensor Least Squares Sketching with Leverage Scores

Thm: Using this sampling probability yields (1+ϵ) accuracy w.h.p. with number of rows

$$s = O(r^d \log(r/\delta)/\epsilon^2)$$

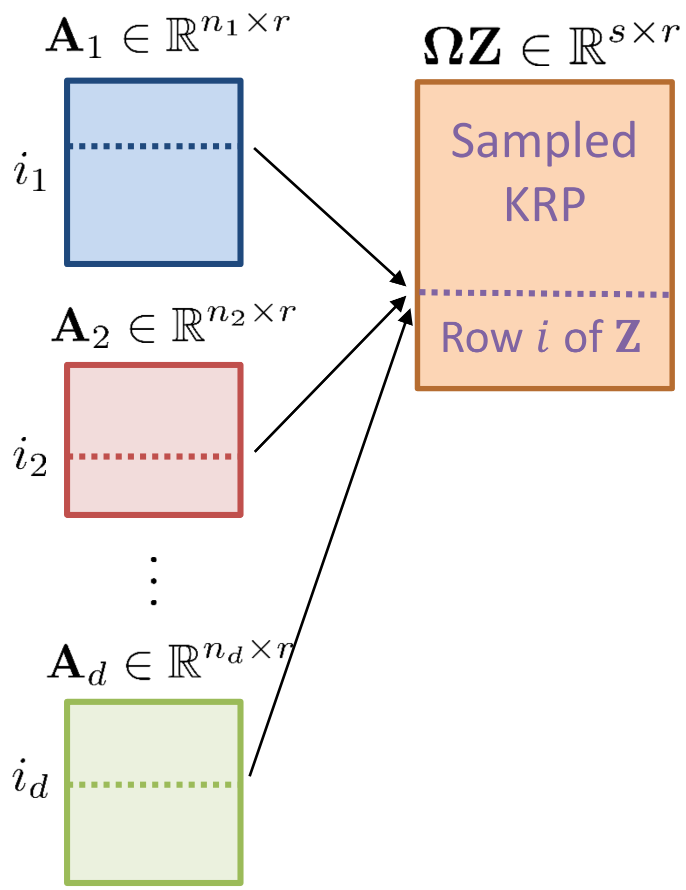
1-1 Correspondence between *linear index and multi index:*

$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

But still don't want to consider all N possible combinations corresponding to all rows of $\mathbf{Z}$!

Ingredient #5: Randomly Sample Rows of KRP Using Implicit Leverage Score Bounds

KRP: $\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_1$



Probability of sampling row i:

$$p_i \equiv \frac{1}{r^d} \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

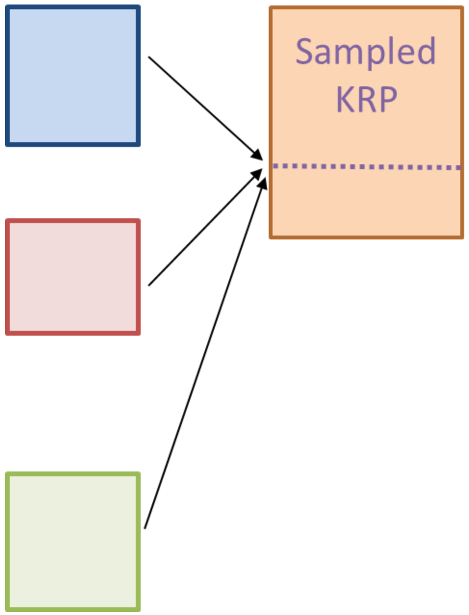
- Recall our goal: Pick s random indices ξ_j such that $P(\xi_j = i) = p_i$
- For j-th sample for $j = 1, \ldots, s$:
  - Sample one row from each factor matrix such that $\mathrm{Prob}(\text{row } i_k) = \ell_{i_k}(\mathbf{A}_k)/r$
  - Set $\xi_j = i$,where $P(\xi_j = i) = p_i$
  - Compute Hadamard products of corresponding rows of factor matrices
  - Weight by $1/\sqrt{s p_i}$
- Never computes…
  - Matrix $\mathbf{Z}$ nor its leverage scores
  - Weight matrix $\mathbf{\Omega}$
- Computing factor matrix leverage scores costs only $O(r^2 \sum_k n_k)$
  - Versus $O(Nr^2)$ for computing leverage scores from $\mathbf{Z}$

1-1 Correspondence between *linear index and multi index:*

$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

Ingredient #6: Combine Repeated Rows

Problem: Concentrated sampling probabilities identify a few key rows *but* can lead to *many* repeats!

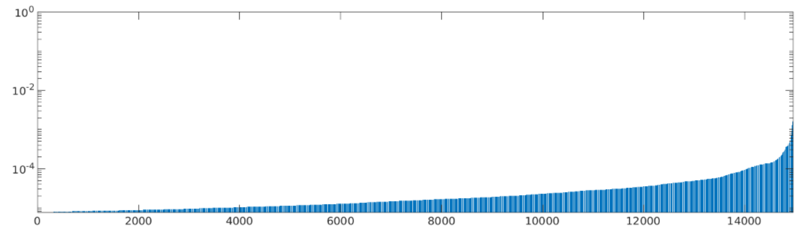


Least Squares Problems from Real-world Tensor Data Sets

Combining repeat rows ⇒ 2-20X speedup

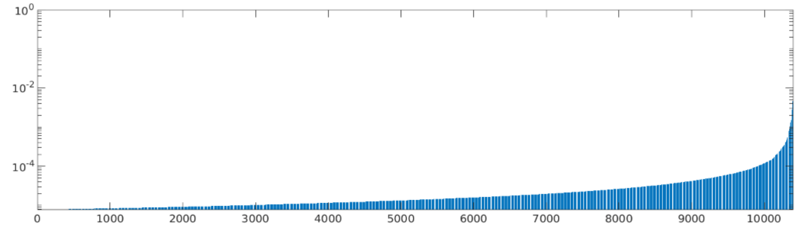
Example 1: $N = 3.2\text{e}12, s = 2^{17}, \tau = \frac{1}{s} = 8\text{e-}6$

$\mathcal{D} = \{i : p_i > \tau\}, |\mathcal{D}| \approx 15000, \sum_{i \in \mathcal{D}} p_i = 0.51$



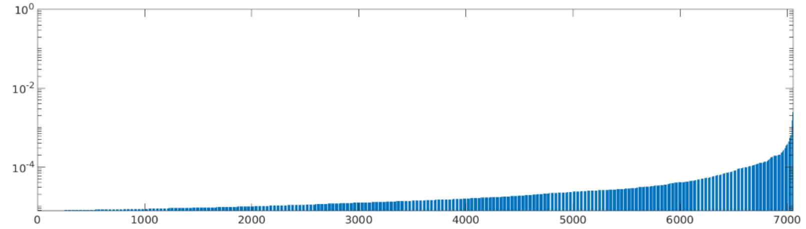
Example 2: $N = 8.7\text{e}12, s = 2^{17}, \tau = \frac{1}{s} = 8\text{e-}6$

$\mathcal{D} = \{i : p_i > \tau\}, |\mathcal{D}| \approx 10000, \sum_{i \in \mathcal{D}} p_i = 0.41$



Example 3: $N = 8.6\text{e}12, s = 2^{17}, \tau = \frac{1}{s} = 8\text{e-}6$

$\mathcal{D} = \{i : p_i > \tau\}, |\mathcal{D}| \approx 7000, \sum_{i \in \mathcal{D}} p_i = 0.25$



Ingredient #7: Hybrid Deterministic and Randomly-Sampled Rows

Deterministic Rows

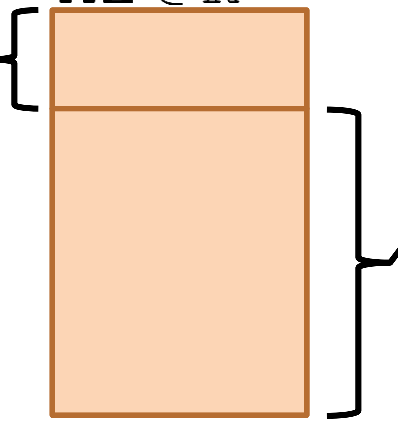
$$\mathcal{D}_\tau = \{\, i \in [N] \mid p_i \geq \tau \,\}$$

$$s_{\text{det}} = |\mathcal{D}_\tau|$$

$$p_{\text{det}} = \sum_{i \in \mathcal{D}_\tau} p_i$$

for $i \in \mathcal{D}_\tau$ **do**
  add row $\mathbf{A}_1(i_1,:) * \cdots * \mathbf{A}_d(i_d,:)$
end for

$\mathbf{\Omega Z} \in \mathbb{R}^{s\times r}$



$$p_i \equiv \frac{1}{r^d} \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

Random Rows

$$s_{\text{rnd}} = s - s_{\text{det}}$$

for $j = 1 \ldots, s_{\text{rnd}}$ **do**
  repeat
    for $k = 1 \ldots, d$ **do**
      $i_k \leftarrow \texttt{multi}(\ell(\mathbf{A}_k)/r)$
    end for
  until $i \notin \mathcal{D}_\tau$
  $\omega \leftarrow \sqrt{(1 - p_{\text{det}})/(s_{\text{rnd}}\, p_i)}$
  add row $\omega\,(\mathbf{A}_1(i_1,:) * \cdots * \mathbf{A}_d(i_d,:))$
end for

1-1 Correspondence between *linear index and multi index:*

$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

Ingredient #9: Find All High-Probability Rows without Computing All Probabilities

- Recall

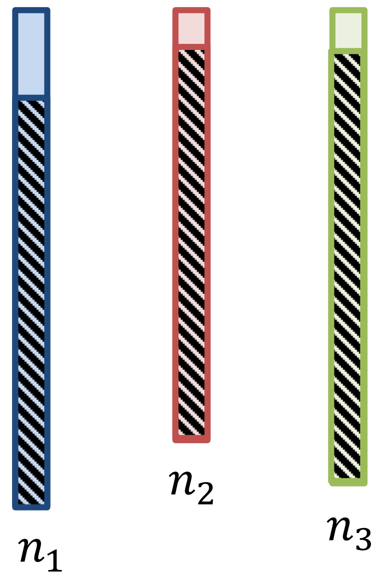
$$p_i \equiv \frac{1}{r^d} \prod_{k=1}^{d} \ell_{i_k}(\mathbf{A}_k)$$

- For given tolerance $\tau > 1/N$, define the set of deterministic rows to include

$$\mathcal{D}_\tau = \{\, i \in [N] \mid p_i \geq \tau \,\}$$

  - Compute *without* computing all p_i values
  - A few high leverage scores means all the others are necessarily low!
  - Use bounding procedure to eliminate most options
  - Compute products of at most a top few leverage scores in each mode

Sorted Leverages Scores (Descending)



1-1 Correspondence between *linear index and multi index:*

$$i \in [N] \Leftrightarrow (i_1, \ldots, i_d) \in [n_1] \otimes \cdots \otimes [n_d]$$

Remember the Original Problem – Need to Sample the Right-Hand Side as Well

Sandia National Laboratories

$$\min_{\mathbf{B}} \|\mathbf{Z}\mathbf{B}^{\intercal} - \mathbf{X}^{\intercal}\|^2$$

$\mathbf{Z} \in \mathbb{R}^{N \times r}$ $\quad \mathbf{B}^{\intercal} \in \mathbb{R}^{r \times n}$ $\quad \mathbf{X}^{\intercal} \in \mathbb{R}^{N \times n}$

Khatri-Rao Product (KRP) Structure

Unknown

$-$

May Be Very Sparse

$$N \gg r, n$$

$$\min_{\mathbf{B}} \|\boldsymbol{\Omega}\mathbf{Z}\mathbf{B}^{\intercal} - \boldsymbol{\Omega}\mathbf{X}^{\intercal}\|^2$$

$\boldsymbol{\Omega}\mathbf{Z} \in \mathbb{R}^{s \times r}$ $\quad \mathbf{B}^{\intercal} \in \mathbb{R}^{r \times n}$ $\quad \boldsymbol{\Omega}\mathbf{X}^{\intercal} \in \mathbb{R}^{s \times n}$

Sampled KRP

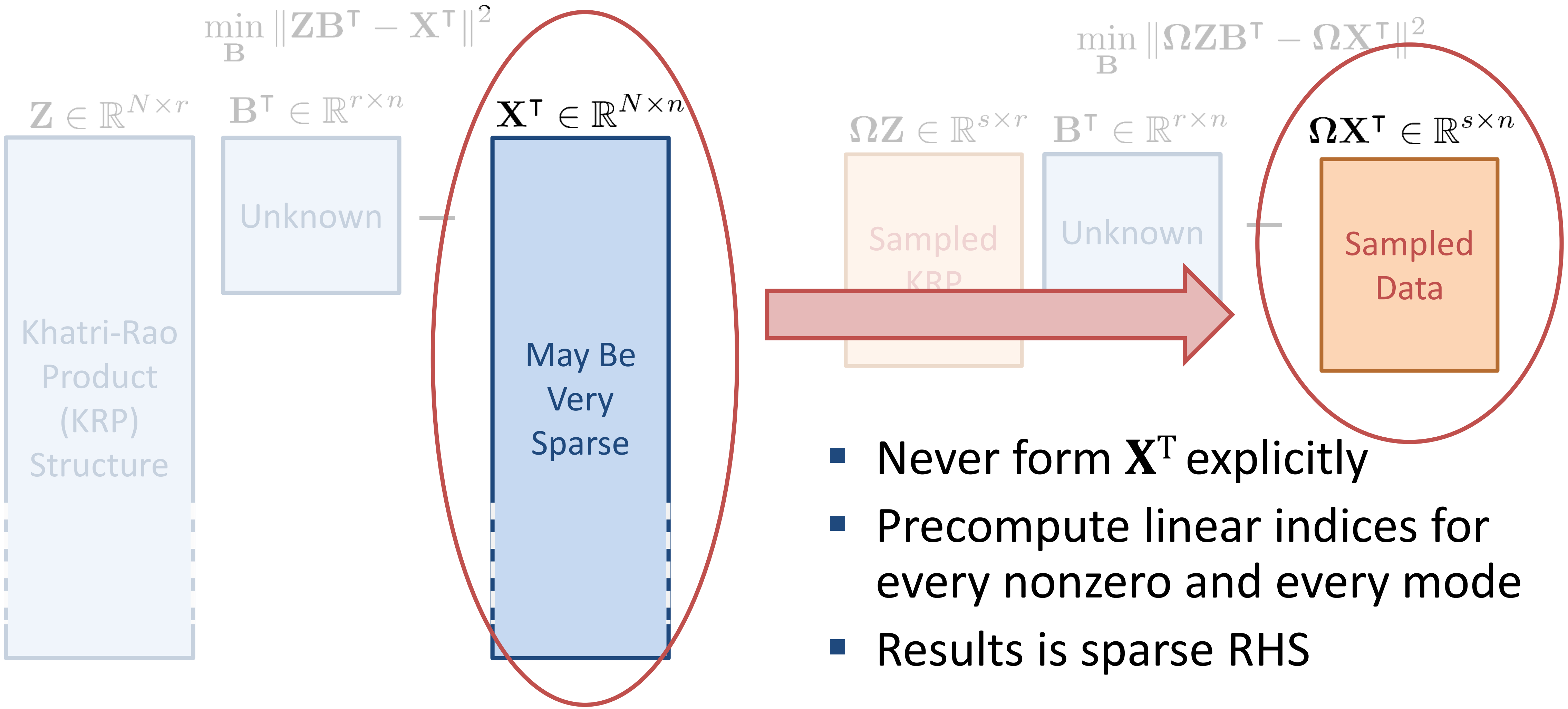
Unknown

$-$

Sampled Data

Complexity reduced from $O(Nrn)$ to $O(sr^2n)$

Ingredient #9: Efficiently Extract RHS from (Sparse) Unfolded Data Tensor



- Never form $\mathbf{X}^{\mathrm{T}}$ explicitly
- Precompute linear indices for every nonzero and every mode
- Results is sparse RHS

Similar in spirit to ideas for dense tensors in Battaglino et al., SIMAX 2018

Numerical Results

Solution Quality as Number of Samples Increase and Hybrid Improvements

Single Least Squares Problem with N = 46M rows, r = 10 columns, n = 183 right-hand sides

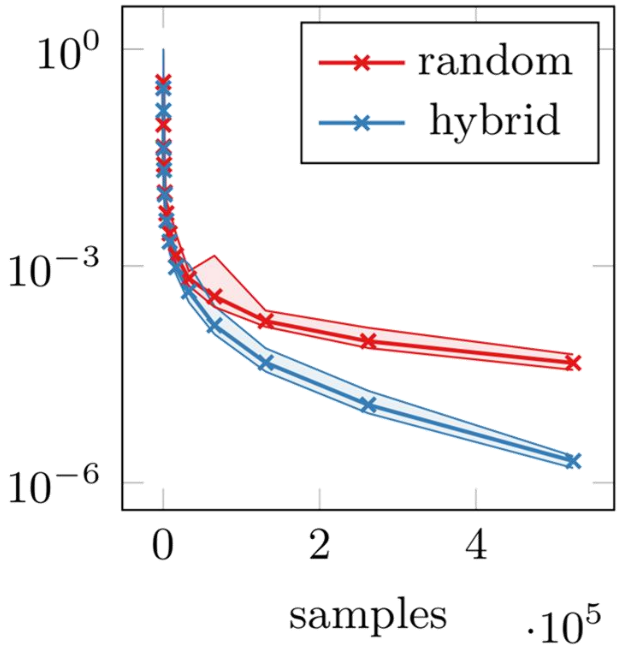
$\mathbf{\Omega Z} \in \mathbb{R}^{s \times r}$ $\mathbf{B}^\intercal \in \mathbb{R}^{r \times n}$ $\mathbf{\Omega X}^\intercal \in \mathbb{R}^{s \times n}$

Sampled KRP | Unknown | − | Sampled Data

$$\tilde{\mathbf{B}}_* \equiv \arg \min_{\mathbf{B} \in \mathbb{R}^r} \|\mathbf{\Omega Z B}^\intercal - \mathbf{\Omega X}^\intercal\|_2^2$$

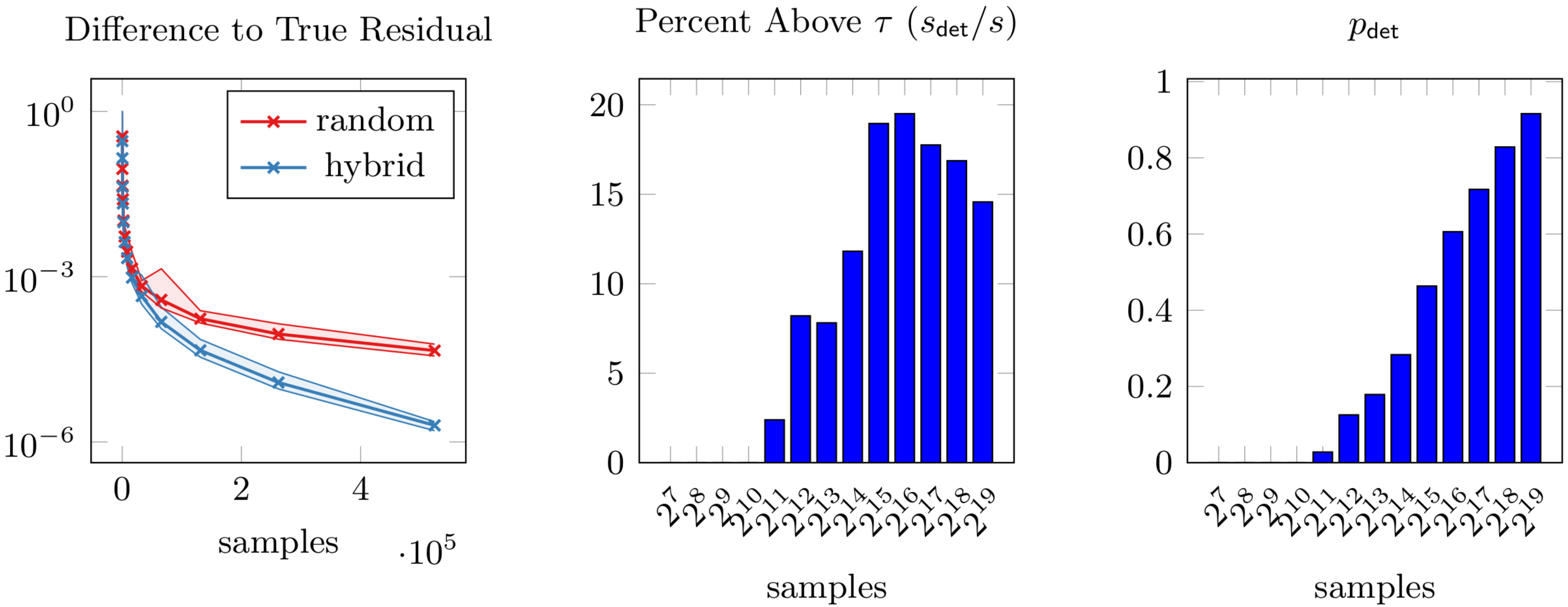
$$\mathbf{B}_* \equiv \arg \min_{\mathbf{B} \in \mathbb{R}^r} \|\mathbf{Z B}^\intercal - \mathbf{X}^\intercal\|_2^2$$

$$\frac{\left| \|\mathbf{Z B}_*^\intercal - \mathbf{X}^\intercal\|_2 - \|\mathbf{Z}\tilde{\mathbf{B}}_*^\intercal - \mathbf{X}^\intercal\|_2 \right|}{\max\{1, \|\mathbf{Z B}_*^\intercal - \mathbf{X}^\intercal\|_2\}}$$

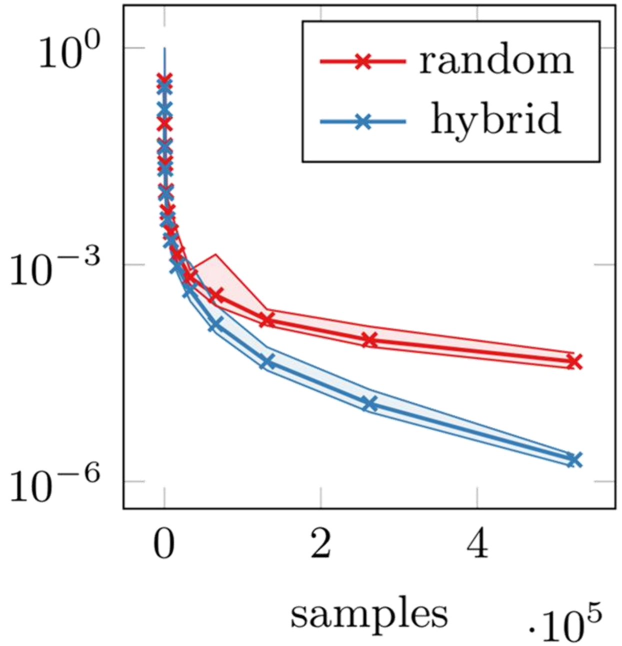


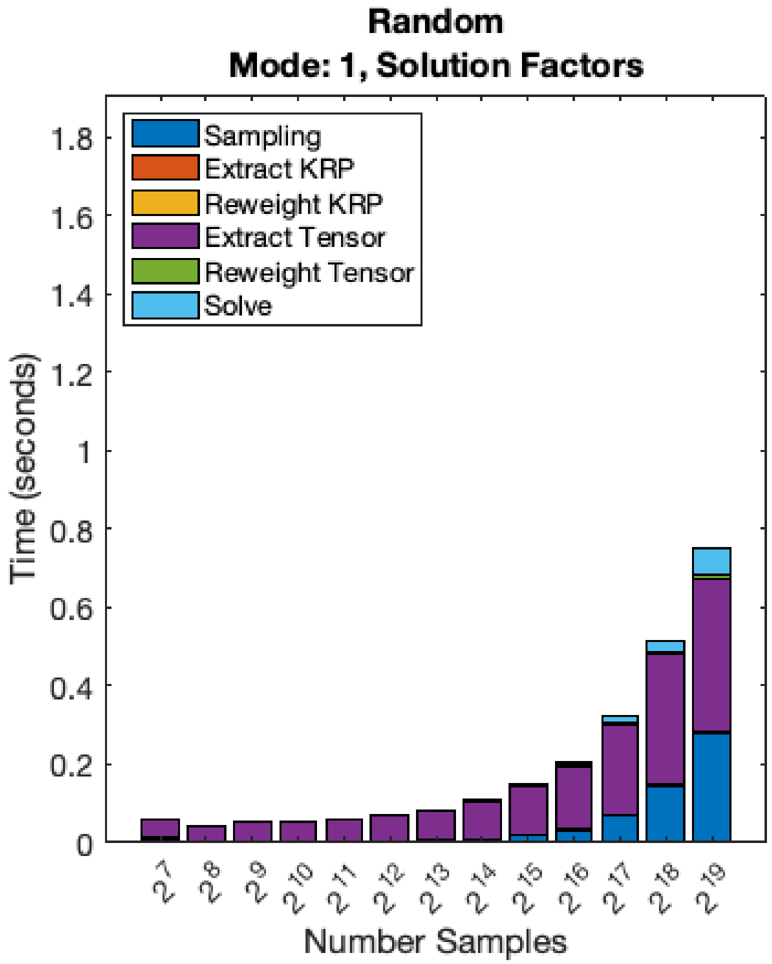
Deterministic Can Account for Substantial Portion of Probability

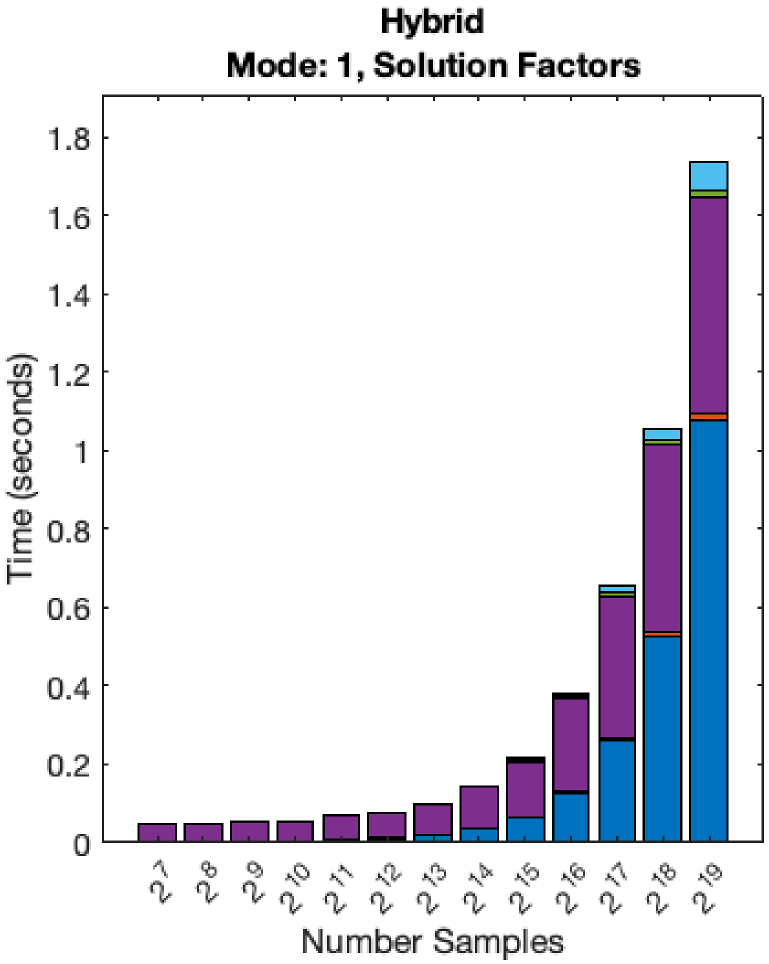
Single Least Squares Problem with N = 46M rows, r = 10 columns, n = 183 right-hand sides



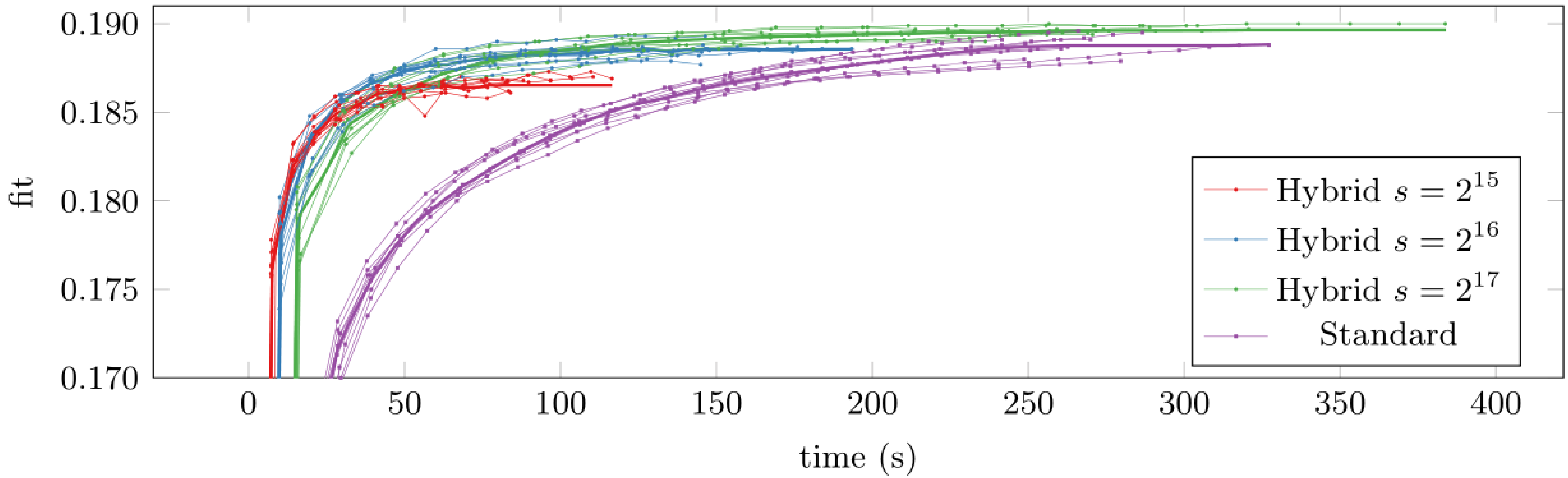
Some Trade-off Between Accuracy and Expense for Deterministic





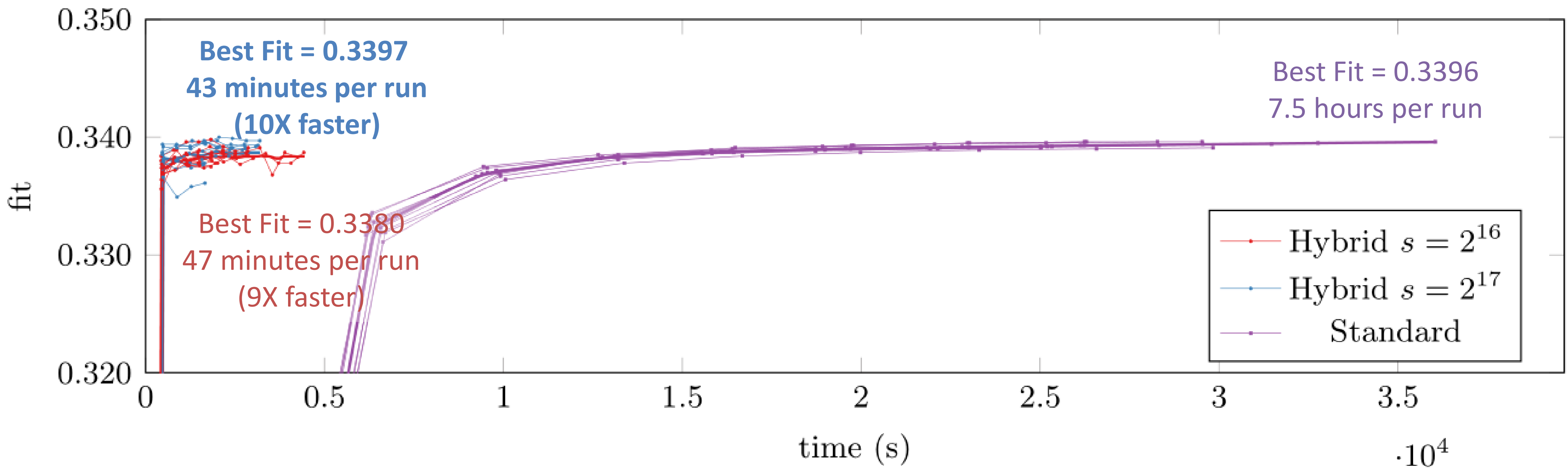


CP-ARLS-LEV (Hybrid) Comparable to CP-ALS (Standard) on Small Uber Problem



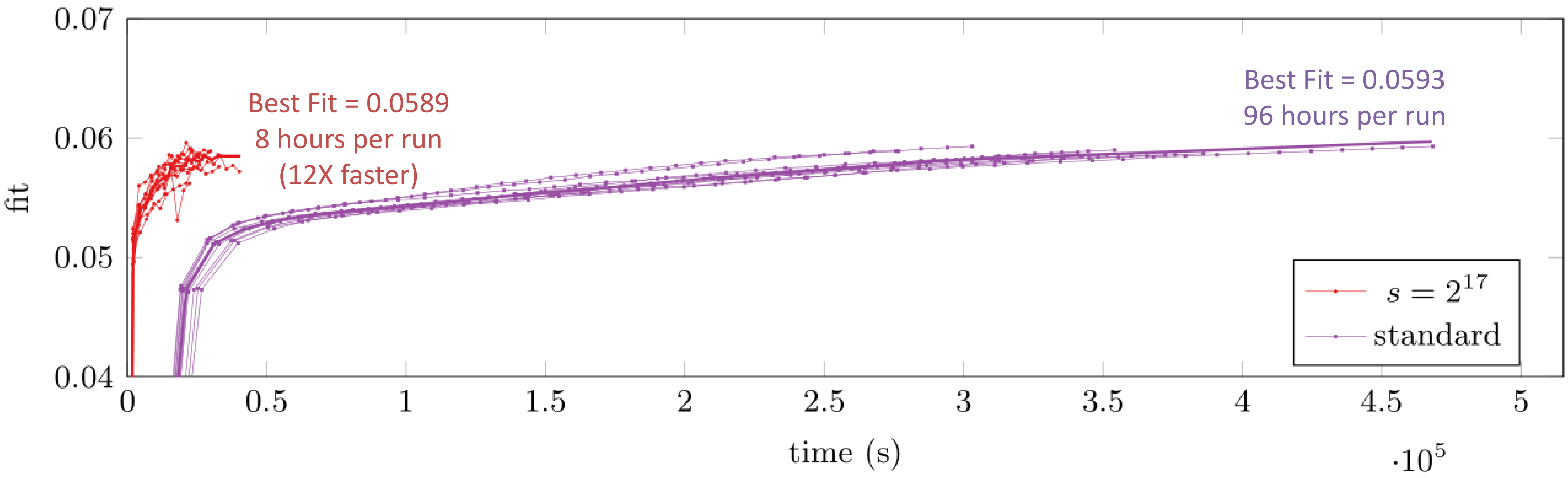
Uber Tensor: 183 x 24 x 1140 x 1717 Uber Tensor with 3M nonzeros (0.038% dense).
Rank r = 25 CP decomposition

Over 9X Speed-up for Amazon Tensor with 1.7 Billion Nonzeros

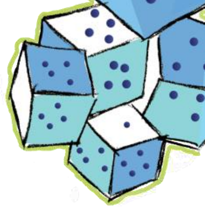


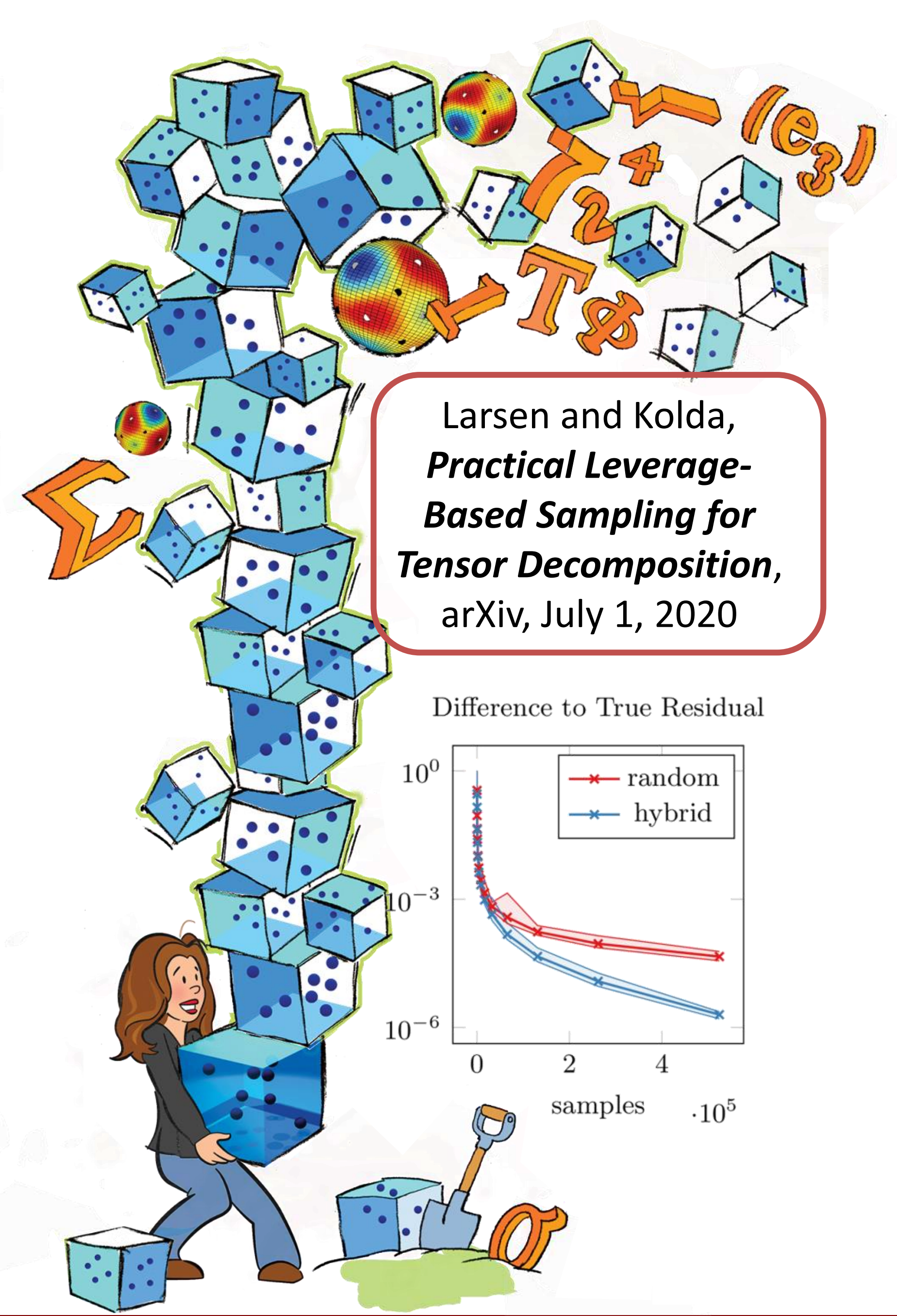
Amazon Tensor: 4.8M x 1.8M x 1.8M Amazon Tensor with 1.7B nonzeros.
Rank r = 25 CP decomposition

Over 12X Speed-up for Reddit Tensor with 4.6 Billion Nonzeros (106 GB)



Amazon Tensor: 8.2M x 0.2M x 8.1M Reddit Tensor with 4.7B nonzeros.
Rank r = 25 CP decomposition





Larsen and Kolda, ***Practical Leverage-Based Sampling for Tensor Decomposition***, arXiv, July 1, 2020

Conclusions & Future Work

- How to make CP tensor decomposition faster for large-scale sparse tensors? Matrix sketching
- How to avoid repeated samples? Combine repeat rows or deterministically include high-probability rows
- How to efficiently sample? Sample independently from each factor matrix to build KRP
- How to extract data for RHS from data tensor? Pre-compute linear indices for tensor fibers
- Overall result: Order-of-magnitude speed-ups
- Many open problems: How to pick # samples (per mode even), deterministic threshold, robust stopping conditions, sampling based on data as well as KRP, parallelization of method, etc.

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